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ABSTRACT

A uniformly distributed normal traction, varying harmonically with time, is applied to the surface of a spherical cavity in an infinite homogeneous, isotropic, elastic solid. The response of the system is found to be similar to that usually associated with damped vibrations of bodies containing reflecting boundaries. An explanation is obtained by comparing the elastic system with a damped, simple oscillator. The same problem is then solved for a cylindrical cavity of circular section and infinite length.

RESPONSE OF AN ELASTIC SOLID TO AN OSCILLATING PRESSURE WITHIN A CAVITY

Diverging Spherical Waves

The body is referred to a system of spherical coordinates r, θ, ϕ, with origin at the center of the cavity. (Fig. 1)

The equation of small motion of an isotropic elastic body[1],

$$\mu \nabla^2 \underset{\sim}{u} + (\lambda+\mu) \nabla \nabla \cdot \underset{\sim}{u} = \rho \frac{\partial^2 \underset{\sim}{u}}{\partial t^2} \tag{1}$$

becomes, on the assumption that $u_r = u_r(r,t)$, $u_\theta = u_\phi = 0$,

$$\frac{\partial}{\partial r}\left[\frac{\partial u_r}{\partial r} + 2\frac{u_r}{r}\right] = \frac{1}{c^2}\frac{\partial^2 u_r}{\partial t^2} \tag{2}$$

where

$$c = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}$$

is the velocity of dilatational waves.

It may be verified by substitution that (2) is identically satisfied by the diverging wave,

$$u_r = \frac{\partial}{\partial r}\left[\frac{1}{r}\chi(r-ct)\right]$$

Using for χ the real part of $Ae^{ik(r-ct)}$, with k an, as yet, undetermined wave number and A a complex constant, we have,

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1. A. E. H. Love, Theory of Elasticity, (Dover Publications, New York, 1944), 4th Edition, p. 278.

$$u_r = A\left[-\frac{1}{r^2} + \frac{ik}{r}\right]e^{ik(r-ct)} \tag{3}$$

The stress components σ_{rr}, $\sigma_{r\theta}$, $\sigma_{r\phi}$ are given by[2]

$$\sigma_{rr} = (\lambda+2\mu)\frac{\partial u_r}{\partial r} + 2\lambda\frac{u_r}{r}, \quad \sigma_{r\theta} = \sigma_{r\phi} = 0. \tag{4}$$

Hence

$$\sigma_{rr} = 4\mu A\left[\frac{1}{r^3} - \beta\frac{k^2}{r} - \frac{ki}{r^2}\right]e^{ik(r-ct)} \tag{5}$$

where

$$\beta = \frac{\lambda+2\mu}{4\mu} = \frac{1-\sigma}{2(1-2\sigma)}$$

and σ is Poisson's ratio.

Forced Spherical Motion

If the applied force is

$$\sigma_{rr}\Big]_{r=a} = Pe^{-i\omega t},$$

we must have

$$|A| = \frac{P}{4\mu}\left[\left(\frac{1}{a^3} - \beta\frac{k^2}{a}\right)^2 + \left(\frac{k}{a^2}\right)^2\right]^{-1/2} \tag{6}$$

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2. Love, p. 142.

and

$$k = \frac{\omega}{c}.$$

The condition of vanishing traction for $r/a \to \infty$ is satisfied.

From (3) and (6) the amplitude of u_r on $r = a$ is given by

$$|u_r|_{r=a} = \frac{aP}{4\mu}\left[\frac{1+p^2}{(1-\beta p^2)^2 + p^2}\right]^{1/2} \tag{7}$$

where the non-dimensional quantity $p = \frac{a\omega}{c}$ contains the effects of the forcing frequency, the radius of the cavity and the physical properties of the material.

To obtain the radial displacement of points on the spherical surface when it is subjected to a uniformly distributed static pressure P, it is only necessary to set $p = 0$ in (7), whence

$$u_{static}\Big]_{r=a} = \frac{aP}{4\mu}. \tag{8}$$

The displacement amplification factor is given by

$$\frac{|u_r|_{r=a}}{u_{st.}\Big]_{r=a}} = \left[\frac{1+p^2}{(1-\beta p^2)^2 + p^2}\right]^{1/2} \tag{9}$$

Amplification factor versus p is plotted for three values of Poisson's ratio in Fig. 2.

The "critical" value of p, i.e., the value of p which produces maximum amplification, is given by

$$p_c = \left[\sqrt{1+2\beta^{-1}} - 1\right]^{1/2}.$$

The corresponding critical value of the forcing frequency ω is

$$\omega_c = \frac{c}{a} p_c = \frac{c}{a}\left[\sqrt{1+2\beta^{-1}} - 1\right]^{1/2}. \tag{10}$$

The steady-state response of the system is seen to be of a type usually associated with damped motions of bodies containing reflecting boundaries. This "resonant" behavior of the infinite solid may be studied further by considering the free vibrations of the system.

Free Spherical Motion

For free motion of the body, σ_{rr} must vanish on $r = a$. Hence, from (5)

$$\frac{1}{a^3} - \beta\frac{k^2}{a} - \frac{ki}{a^2} = 0,$$

so that

$$k = \pm k_1 - k_2 i \tag{11}$$

where

$$k_1 = \frac{(4\beta - 1)^{1/2}}{2\beta a}, \qquad k_2 = \frac{1}{2\beta a}. \tag{12}$$

(Note that k_1 and k_2 are positive, real quantities since $\beta \geq 1/3$ for $-1 \leq \sigma \leq 1/2$. Also, we may replace $\pm k_1$, by k_1 as the minus sign merely introduces a phase change in the displacements.)

We must ascertain whether or not the complex value of k given in (11) gives rise, upon substitution into (3) and (5), to displacements and tractions which vanish at infinity. We observe that

$$A\left[-\frac{1}{r^2} + \frac{ik}{r}\right]e^{ik(r-ct)} = A' r^{-1/2} H^{(2)}_{3/2}(kr) e^{-ikct}$$

where A' is a complex constant and $H^{(2)}_{3/2}(kr)$ is a Hankel function of the second kind. These functions vanish at infinity when the imaginary part of their argument is negative[3], as it is in this case, according to (11).

The motion is a diverging wave. A complex circular frequency of particle motion, Ω, may be defined to characterize the time dependence of the displacements,

$$\Omega = kc = c(k_1 - k_2 i). \tag{13}$$

We note, for future reference, that the relationship between the critical frequency of forced motion, ω_c, and the complex frequency associated with free motion, Ω, is given from (10), (12) and (13) by

$$\omega_c = \frac{2\beta\Omega\left[\sqrt{1+2\beta^{-1}}-1\right]^{1/2}}{\sqrt{4\beta-1}\ -i} \tag{14}$$

<u>Simple Oscillator</u>

Consider now the response of a Voigt Element. A spring (spring constant $= S$, dash-pot (coefficient of viscous damping $= \eta$) and mass (m) are arranged as shown in Fig. 3.

The free vibrations of m are characterized by the complex frequency

$$\Omega^* = \omega_0^*\left[\sqrt{1-\alpha^2} - i\alpha\right] \tag{15}$$

where

$\omega_0^* = \sqrt{\frac{S}{m}}$ is the natural frequency of undamped motion and

$\alpha = 2\omega_0 m$ is the fraction of critical damping.

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3. Jahnke and Emde, Tables of Functions, (Dover Publications, New York, 1945), 4th Edition, p. 133.

If the support N is forced to perform harmonic oscillations with a circular frequency, ω^*, the ratio of the amplitude of the steady-state motion of m to the amplitude of the forced motion of N is given by the amplification factor

$$A^* = \left[\frac{1 + \left[2\alpha \frac{\omega^*}{\omega_0^*}\right]^2}{\left[1 - \left(\frac{\omega^*}{\omega_0^*}\right)^2\right]^2 + \left[2\alpha \frac{\omega^*}{\omega_0^*}\right]^2} \right]^{1/2} \tag{16}$$

Maximum amplification occurs when the forcing frequency, ω^*, takes on the critical value

$$\omega_c^* = \frac{\omega_0^*}{2\alpha}\left[\sqrt{1+8\alpha^2} - 1\right]^{1/2}$$

or, expressed in terms of Ω^*, from (15),

$$\omega_c^* = \frac{\Omega^*}{2\alpha} \frac{\left[\sqrt{1+8\alpha^2} - 1\right]^{1/2}}{\sqrt{1-\alpha^2} - i\alpha}. \tag{17}$$

The resonant response of this system is attributed to the fact that the forced motion excites a possible mode of free motion. Maximum response is obtained when the frequencies of the respective motions are related as in (17).

Comparison between Simple Oscillator and Spherical Motion of Elastic Continuum

Comparison of (9) and (16) shows that the amplification factors for the infinite solid and the Voigt Element have the same form. It is thus possible to identify the parameters of one system with their counterparts in the other, and we may, for instance, discuss the infinite solid in terms of the oscillator if we change

$$p \to 2\alpha \frac{\omega^*}{\omega_0^*}$$

$$\beta \to \frac{1}{4\alpha^2} \tag{18}$$

Further, the relation between critical forcing frequency and natural frequency is the same in the two systems. Performing in (14) the changes indicated by (18), we have,

$$\omega_c \to \frac{\Omega\left[\sqrt{1+8\alpha^2}-1\right]^{1/2}}{2\alpha\sqrt{1-\alpha^2}-i\alpha},$$

and this is identical with the corresponding relation for the oscillator as given by (17).

The resonant response of the infinite body may then be interpreted as follows: The system is capable of performing a free, radial, oscillatory motion which has a transient character. Damping of the motion is due to the fact that the energy stored in the system at any instant is thereafter propagated outward radially, there being no reflecting boundaries. The free motion has associated with it a definite frequency of undamped particle oscillation, ω_0, given by the modulus of the complex frequency Ω, that is, from (12) and (13),

$$\omega_0 = \frac{2}{a}\sqrt{\frac{\mu}{\rho}}.$$

This may be compared with the natural frequency of radial vibration of a spherical shell[4] of radius a:

$$\frac{2}{a}\sqrt{\frac{\mu}{\rho}\,\frac{1+\sigma}{1-\sigma}}.$$

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4. Love, p. 287.

Thus, the undamped natural frequency of an infinite body with a spherical cavity of radius a is the same as the frequency of a spherical shell of radius $a\sqrt{(1+\sigma)/(1-\sigma)}$. When harmonic surface tractions are applied across the surface of the spherical cavity, the forced motion, thereby induced, excites the free motion and resonance occurs.

Diverging Cylindrical Waves

The body is referred to a system of cylindrical coordinates r, θ, z, the z-axis being coincident with the axis of the cylinder. (Fig. 4)

Assuming $u_r = u_r(r,t)$, $u_\theta = u_z = 0$, the equation of motion (1) reduces to

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right] = \frac{1}{c^2}\frac{\partial^2 u_r}{\partial t^2} \tag{19}$$

The solution of (19) which corresponds to diverging waves[5], is

$$u_r = C\left[J_1(kr) + iY_1(kr)\right]e^{-ikct} \tag{20}$$

where k is an, as yet, undetermined wave number, C is a complex constant, and J_1 and Y_1 are Bessel's functions of order one, of the first and second kind respectively.

The stress components σ_{rr}, $\sigma_{r\theta}$, σ_{rz} are given by[6]

$$\sigma_{rr} = \frac{\lambda}{r}\frac{\partial}{\partial r}(r u_r) + 2\mu\frac{\partial u_r}{\partial r}, \quad \sigma_{r\theta} = \sigma_{rz} = 0. \tag{21}$$

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5. Lamb, Hydrodynamics, (Dover Publications, New York, 1932), 6th Edition, p. 524.
6. Love, p. 288.

Substituting (20) into (21),

$$\sigma_{rr} = \frac{C}{r}\Big\{[\lambda J_1(kr) + (\lambda+2\mu) rk J_1'(kr)] + i[\lambda Y_1(kr) + (\lambda+2\mu) rk Y_1'(kr)]\Big\} e^{-ikct} \qquad (22)$$

where primes denote differentiation with respect to the argument.

Forced Cylindrical Motion

The condition

$$\sigma_{rr}\Big]_{r=a} = P e^{-i\omega t}$$

requires that

$$\left.\begin{aligned} |C| &= aP\Big\{[\lambda J_1(ka) + (\lambda+2\mu) ak J_1'(ka)]^2 + [\lambda Y_1(ka) + (\lambda+2\mu) ak Y_1'(ka)]^2\Big\}^{-1/2} \\ &\text{and} \quad k = \frac{\omega}{c} \end{aligned}\right\} \qquad (23)$$

The condition of vanishing traction for $r/a \to \infty$ is satisfied.

From (20) and (23), the amplitude of u_r on $r = a$ is given by

$$|u_r|_{r=a} = \frac{aP}{2\mu}\left[\frac{J_1^2(p) + Y_1^2(p)}{[J_1(p) - \beta' p J_0(p)]^2 + [Y_1(p) - \beta' p Y_0(p)]^2}\right]^{1/2} \qquad (24)$$

where

$$p = \frac{a\omega}{c}, \quad \beta' = \frac{\lambda+2\mu}{\mu} = \frac{2(1-\sigma)}{1-2\sigma}$$

and J_0, Y_0 are Bessel's functions of order zero.

The radial displacement of points on the cylindrical surface, when it is subjected to a uniformly distributed static pressure P, is obtained at the limit of (24) as $p \to 0$. Thus,

$$u_{static}\Big]_{r=a} = \frac{aP}{2\mu}.$$

The displacement amplification factor is given by

$$\frac{|u_r|_{r=a}}{u_{st}\Big]_{r=a}} = \left[\frac{J_1^2(p) + Y_1^2(p)}{[J_1(p) - \beta' p J_0(p)]^2 + [Y_1(p) - \beta' p Y_0(p)]^2}\right]^{1/2}$$

Amplification factor versus p is plotted for three values of Poisson's ratio, in Fig. 5. Here, again, it is seen that the amplification factor has the character of that of a simple oscillator. However, in this case, explicit formulas cannot be obtained for the undamped natural frequency and diameter of the equivalent cylindrical shell.

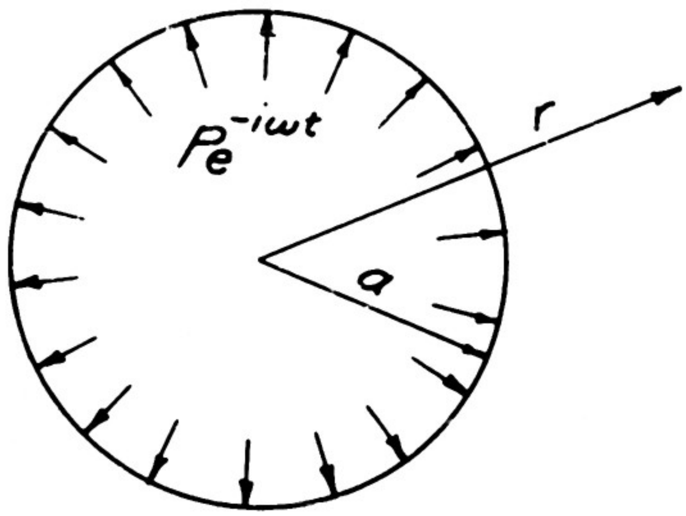

Fig. 1: Spherical Cavity

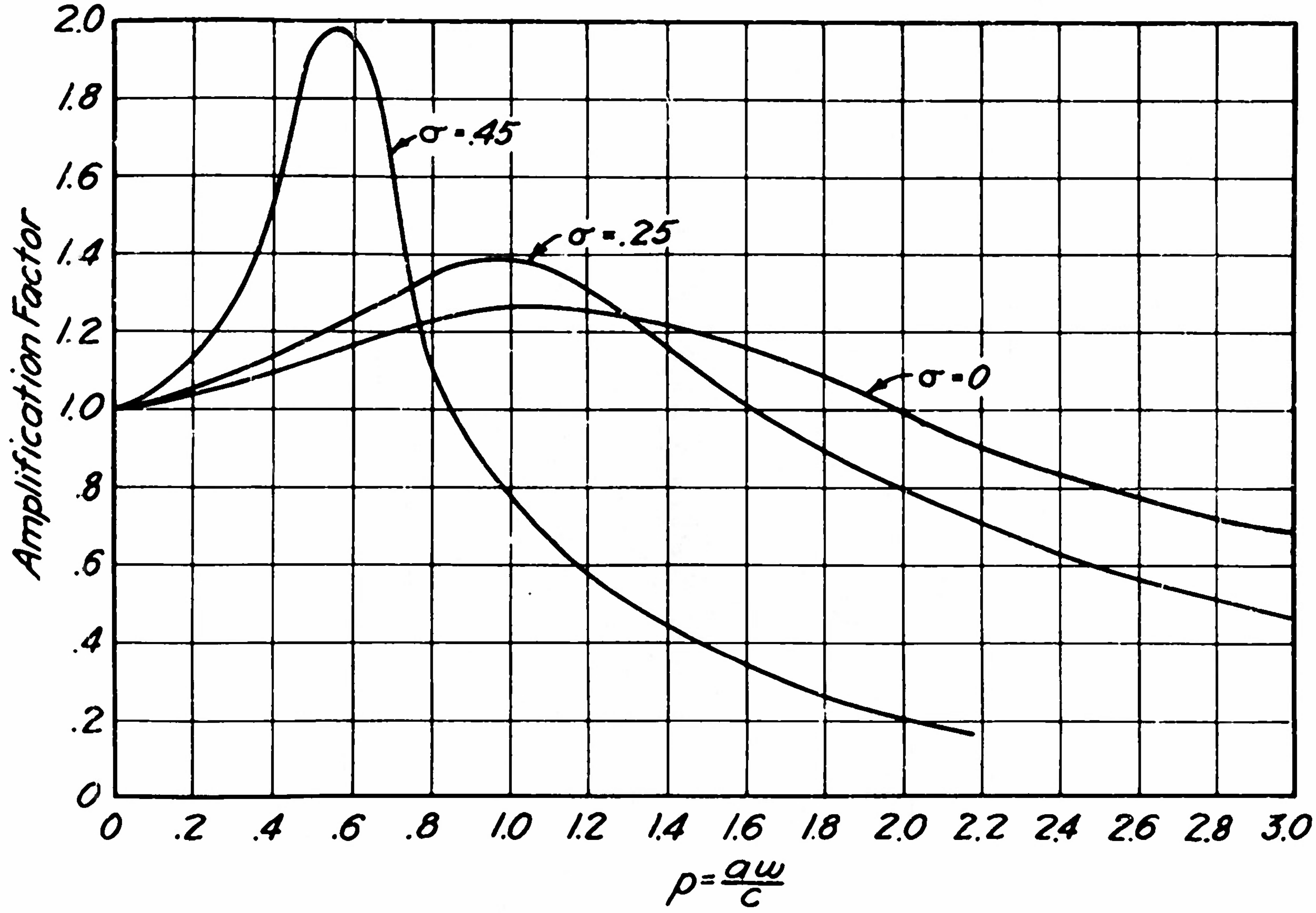

Fig. 2: Amplification Factor - Spherical Cavity

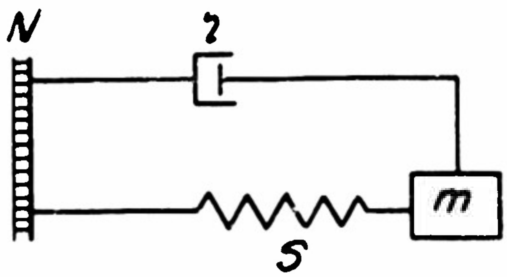

Fig. 3: Voigt Element

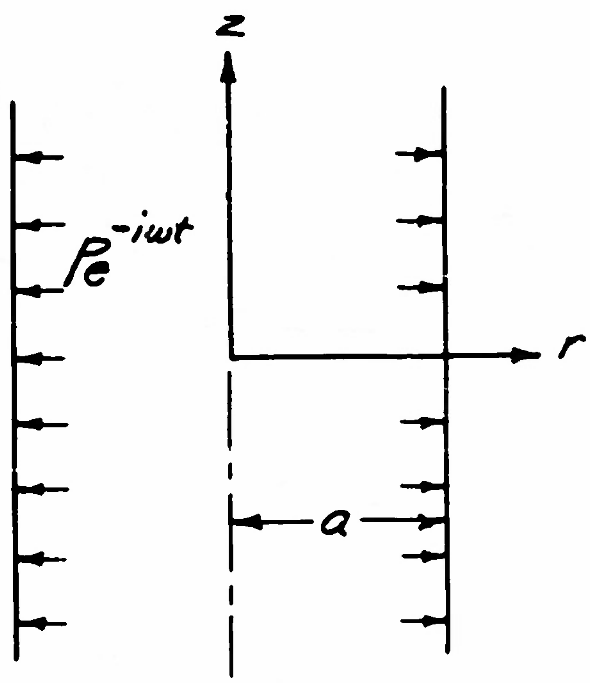

Fig. 4: Cylindrical Cavity

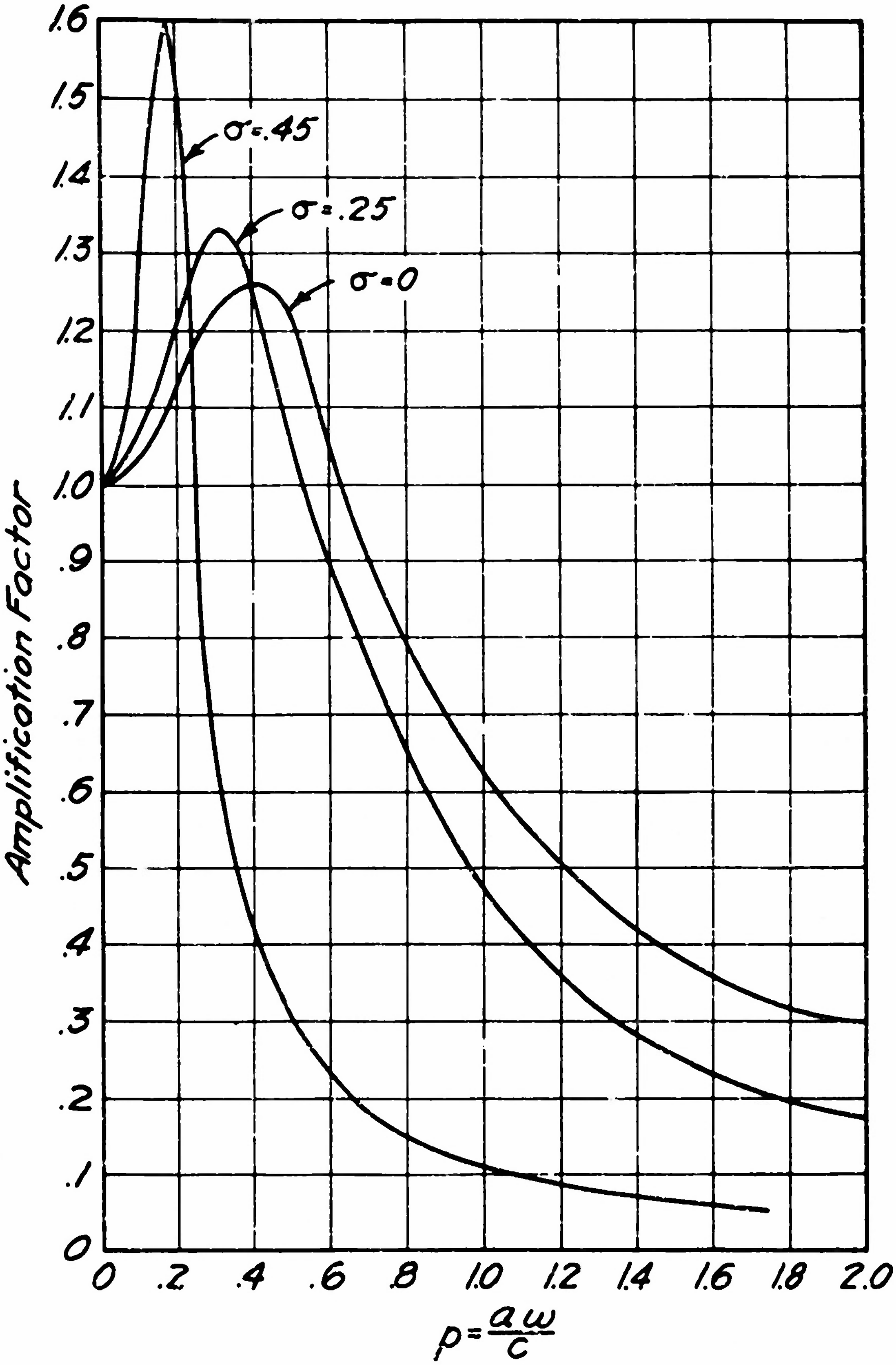

Fig. 5: Amplification Factor - Cylindrical Cavity